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# The Pope's Rhinoceros and Quantum Mechanics

Michael Gulas
gulasm@bgsu.edu

# The Pope's Rhinoceros and Quantum Mechanics

Michael Gulas

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Dr. Steven Seubert Dept. of Mathematics and Statistics, Advisor
Dr. Marco Nardone Dept. of Physics and Astronomy, Advisor

# Contents

*Chapter 1*

# Abstract

In this project, I unravel various mathematical milestones and relate them to the human experience. I show and explain the solution to the Tautochrone, a popular variation on the Brachistochrone, which details a major battle between Leibniz and Newton for the title of inventor of Calculus. One way to solve the Tautochrone is using Laplace Transforms; in this project I expound on common functions that get transformed and how those can be used to solve the Tautochrone. I then connect the solution of the Tautochrone to clock making. From this understanding of clocks, I examine mankind's understanding of time and discuss the Longitude Problem and time unification. Finally, I venture into modern physics where our understanding of time and physics is again shifted. Overall, I bridge the gap between Classical Mechanics & Quantum Mechanics and our understanding of time.

*Chapter 2*

# Classical Mechanics

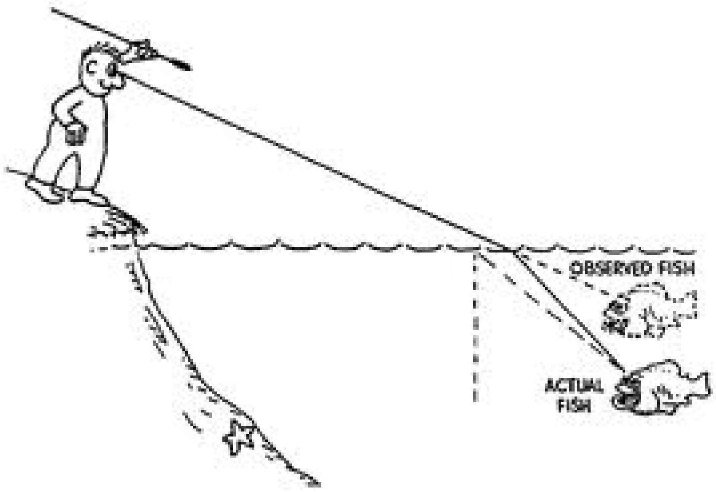
In this section, I discuss the Brachistochrone, an important mathematics problem first given to Newton as a challenge from Leibniz. I then introduce Laplace Transforms, a powerful tool to solve differential equations. I then do a few examples of solving differential equations. Having gained an understanding of Laplace Transforms and Convolution, I then conquer a popular variation of the Brachistochrone, the Tautochrone. After that, I derive the ideal shape of a water clock, a Clepsydra. I then discuss another powerful math tool, the Calculus of Variations. Lastly, I revisit the solution to the Brachistochrone using this method.

## 2.1 Brachistochrone

This first problem details one of the battles of Newton vs. Leibniz regarding who was the true inventor of the Calculus. In 1696, Newton, age 53, retired as a Professor at Cambridge to work at the Mint in London. Shortly after, he received a letter detailing 2 challenge problems from Bernoulli. The original problem goes as the following: "to determine the curved line joining two given points, situated at different distances from the horizontal and not in the same vertical line, along which a mobile body, running down by its own weight and starting to move from the upper point, will descend most quickly to the lowest point." In a bit more modern vernacular this might go as the following: "Given two points in the coordinate $xy$-plane, $A$ and $B$, what is the shape of a smooth curve connecting the two such that a bead would without friction roll down in the shortest amount of time?" Bernoulli coined this famous problem the Brachistochrone, stemming from Greek roots; "brachistos" meaning shortest and "chronos" meaning time- together we have shortest time.

Here we follow the ideology of Leibniz and Bernoulli for the derivation of the solution to the Brachistochrone problem. We need to minimize the time, so the time, $t$, of the bead is given by $t = \frac{d}{v}$ where $d$ is the length of the curve and $v$ is the velocity of the bead. However, this is not enough to go forward with.

First, we need to understand Snell's Law, a physical law regarding optics. Say you are hunting fish with a spear, you will throw your spear from 6 feet in the air at a fish that is 20 feet away from you, 4 feet under the water.




Now if you instead imagine the path of light from your eyes to the fish, which is not simply a straight line, the path of the light was refracted at the surface of the water. We can determine the time it would take by simply finding the distance the light traveled in air divided by the velocity of light in air, $v_{\text{a}}$, plus the distance the light traveled in water divided by the velocity of light in water, $v_{\text{w}}$.

$$t = \frac{\text{distance traveled in air}}{\text{velocity of light in air}} + \frac{\text{distance traveled in water}}{\text{velocity if light in water}}$$

I define $\theta_{in}$ to be angle made between your eyes to the water and the line normal to the surface of the water and $\theta_{out}$ as the angle made from the fish to the surface of the water and the normal line. We can get the distances traveled using the Pythagorean Theorem.

$$t = \frac{d}{v} = \frac{\sqrt{(x^2+36)}}{v_{\text{a}}} + \frac{\sqrt{16+(20-x)^2}}{v_{\text{w}}}$$

$$t' = \frac{x}{v_{\text{a}}\sqrt{x^2+36)}} - \frac{(20-x)}{v_{\text{w}}\sqrt{16+(20-x)^2}}$$

Now using trigonometry we can see that two sine expressions are in $t'$.

$$t' = \frac{\sin(\theta_{in})}{v_{\text{a}}} - \frac{\sin(\theta_{out})}{v_{\text{w}}}$$

Trying to minimize this function we require that

$$0 = \frac{\sin(\theta_{in})}{v_{\text{a}}} - \frac{\sin(\theta_{out})}{v_{\text{w}}}$$

$$\frac{\sin(\theta_{in})}{v_{\text{a}}} = \frac{\sin(\theta_{out})}{v_{\text{w}}}.$$

This is Snell's Law.

Second, we need to imagine that at each instance the bead is being refracted as it travels down its curve. We need to imagine that it is being refracted an infinite amount of times through infinitely thin layers. At each boundary we are imagining the bead really behaves like light and is trying to minimize the time traveled on each length of the curve at every instant. Let $\alpha$ be the complement to $\theta$. Due to Snell's Law, at each layer of refraction we thus require that:

$$constant = \frac{\sin(\theta)}{v} = \frac{\cos(\alpha)}{v} = \frac{1}{v\sec(\alpha)}.$$

Note that $\tan \alpha$ is the derivative of our function. Using some trigonometric identities to reduce the above and calling that constant $c$,

$$c = \frac{1}{v \sec(\alpha)} = \frac{1}{v\sqrt{1+\tan^2(\alpha)}} = \frac{1}{v\sqrt{1+(\frac{dy}{dx})^2}}.$$

Inverting each side and then squaring we obtain

$$c = v^2\big(1+(\frac{dy}{dx})^2\big) = v^2\big(1+(y')^2\big).$$

Using the kinetic and potential energy of an object in Newtonian mechanics we know that total mechanical energy is conserved, and for an appropriate coordinate system (our $xy$-plane) we can write,

$$E_{Kinetic} = \frac{1}{2}mv^2 = E_{Potential} = mgy.$$

Solving for $v^2$:

$$v^2 = 2gy.$$

Notice that the velocity does not depend on the mass of the particle and thus the curve does not depend on the particle's mass. Plugging in what we know:

$$c = 2gy(1+(y')^2)$$

Solving for $y'$:

$$\sqrt{\frac{c}{y}-1} = y'.$$

The differential equation is separable. To solve it we make a trigonometric substitution here (See Appendix A.2).

$$y = c\sin^2(\theta)$$

Solving:

$$x = \frac{c}{2}(2\theta - \sin(2\theta))$$

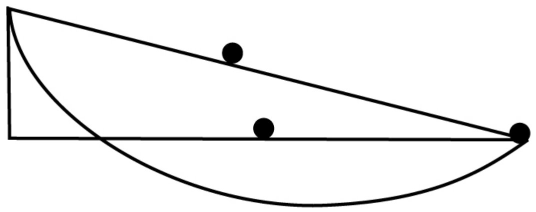
$$y = \frac{c}{2}(1-\cos(2\theta)).$$

Finally, making a small substitution $t = 2\theta$ and absorbing the $c/2$ into $c$.

$$x = c(t-\sin(t))$$

$$y = c(1-\cos(t))$$

Now let us examine Newton's method of solving it. He was able to deduce the solution was a cycloid which looks like the figure below. Observe that the straight line is the shortest distance, but the curved cycloid path is the path of shortest time.



## 2.2 Laplace Transforms

Laplace transforms are useful for many reasons; they happen to be useful to me currently to solve differential equations that I could not solve using normal methods. Using Laplace transforms I can simplify many ordinary differential equations (ODE's) into algebra or simpler ODE's. The Laplace Transform of $f(t)$ is

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t)\ dt = \mathscr{L}\{f(t)\}.}$$

Let us take the Laplace Transform of $f(t) = 1$.

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st} 1\ dt$$

$$\int_0^\infty e^{-st}\ dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = \frac{1}{s}$$

Thus the Laplace transform of $f(t) = 1$ is

$$\mathscr{L}\{1\} = \frac{1}{s}.$$

Let us try something a bit harder, such as $f(t) = e^{at}$, $a$ is a real number with $s - a > 0$.

$$\mathscr{L}\{e^{at}\} = F(s) = \int_0^\infty e^{-st} e^{at}\ dt$$

$$= \int_0^\infty e^{-st+at}\, dt = \int_0^\infty e^{t(a-s)}\, dt$$

$$= \frac{1}{(a-s)} e^{-st+at}\bigg|_0^\infty = \frac{1}{s-a}$$

Thus the Laplace transform of $f(t) = e^{at}$ is

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}.$$

Let us try $f(t) = t^n$ for positive integers n. Let us first check $f(t) = t$.

$$\mathscr{L}\{t\} = F(s) = \int_0^\infty e^{-st} t\, dt$$

Using integration by parts we obtain:

$$\int_0^\infty e^{-st} t\, dt = -\frac{t}{s}e^{-st}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt$$

$$= -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}$$

Thus,

$$\mathscr{L}\{t\} = \frac{1}{s^2}.$$

Likewise we can use integration by parts twice and three times to find respectively,

$$\mathscr{L}\{t^2\} = \frac{2}{s^3}$$

$$\mathscr{L}\{t^3\} = \frac{6}{s^4}.$$

Observe that

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Let us prove this by induction. If we check $f(t) = t$ as our basis step, this is true. Let us consider it true for some $k \geq 1$ that is:

$$\mathscr{L}\{t^k\} = \frac{k!}{s^{k+1}}$$

We want to show that it is true for $k+1$. We want to see that

$$\mathscr{L}\{t^{k+1}\} = \frac{(k+1)!}{s^{k+2}}$$

$$\mathscr{L}\{t^{k+1}\} = \int_0^\infty e^{-st}t^{k+1}\,dt.$$

Still using integration by parts we obtain:

$$-\frac{t^{k+1}}{s}e^{-st}\Big|_0^\infty + \frac{(k+1)}{s}\int_0^\infty e^{-st}t^k\,dt$$

By our induction assumption we know that integral.

$$-\frac{t^{k+1}}{s}e^{-st}\Big|_0^\infty + \frac{(k+1)}{s}\frac{k!}{s^{k+1}}$$

The first parts evaluates to zero and the 2nd part combines to exactly what we wanted.

$$\frac{(k+1)!}{s^{k+2}}$$

Thus by induction

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Let us do the trigonometric function sine. Let $f(t) = \sin(at)$.

$$\mathscr{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)\,dt$$

Let us use integration by parts to obtain

$$\int_0^\infty e^{-st}\sin(at)\,dt = -\frac{\sin(at)}{s}e^{-st}\Big|_0^\infty + \frac{a}{s}\int_0^\infty e^{-st}\cos(at)\,dt.$$

Now using integration by parts again to obtain

$$-\frac{\sin(at)}{s}e^{-st}\bigg|_0^\infty + \frac{a}{s}\left[-\frac{\cos(at)}{s}e^{-st}\bigg|_0^\infty - \frac{a}{s}\int_0^\infty e^{-st}\sin(at)\,dt\right]$$

$$-\frac{\sin(at)}{s}e^{-st}\bigg|_0^\infty - \frac{a\cos(at)}{s^2}\bigg|_0^\infty - \frac{a^2}{s^2}\int_0^\infty e^{-st}\sin(at)\,dt$$

$$\left(1+\frac{a^2}{s^2}\right)\int_0^\infty e^{-st}\sin(at)\,dt = -\frac{\sin(at)}{s}e^{-st} - \frac{a\cos(at)}{s^2}\bigg|_0^\infty$$

$$\int_0^\infty e^{-st}\sin(at)\,dt = \frac{s^2}{s^2+a^2}\left[-\frac{\sin(at)}{s}e^{-st} - \frac{a\cos(at)}{s^2}\bigg|_0^\infty\right]$$

$$\int_0^\infty e^{-st}\sin(at)\,dt = \frac{s^2}{s^2+a^2}\left[(0-0)-(0-\frac{a}{s^2})\right] = \frac{a}{s^2+a^2}.$$

Thus the Laplace transform of $f(t)=\sin(at)$ is

$$\mathscr{L}\{\sin(at)\} = \frac{a}{s^2+a^2}.$$

Let us take the Laplace transform of $f(t)=\cos(at)$.

$$\mathscr{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,dt$$

Let us start with integration by parts again:

$$\int_0^\infty e^{-st}\cos(at)\,dt = \frac{\sin(at)}{a}e^{-st}\bigg|_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin(at)\,dt.$$

But wait, we have already done that integral from earlier, that is the Laplace transform of $\sin(at)$

$$\frac{\sin(at)}{a}e^{-st}\bigg|_0^\infty + \frac{s}{a}\left[\frac{a}{a^2+s^2}\right]$$

The first term wipes out and we are left with

$$\frac{s}{a}\left[\frac{a}{a^2+s^2}\right] = \frac{s}{s^2+a^2}.$$

Thus the Laplace transform of $f(t)=\cos(at)$ is

$$\mathscr{L}\{\cos(at)\} = \frac{s}{s^2+a^2}.$$

Let us consider $f(t)=kg(t)$ with $k$ as an arbitrary constant.

$$\mathscr{L}\{kg(t)\} = \int_0^\infty e^{-st}kg(t)\,dt$$

Because you can take constants out from integrals, you can take them out from Laplace Transforms.

$$\mathscr{L}\{kg(t)\} = k\int_0^\infty e^{-st}g(t)\,dt$$

Thus,

$$\mathscr{L}\{kg(t)\} = kF(s).$$

Let us consider $f(t) = tg(t)$.

$$\mathscr{L}\{tg(t)\} = \int_0^\infty e^{-st} t g(t)\, dt$$

Consider this identity

$$F(s) = \int_0^\infty e^{-st} g(t)\, dt.$$

If I take the derivative with respect to $s$ of both sides I get

$$\frac{dF(s)}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} g(t)\, dt.$$

Now using Leibniz's Rule, I can differentiate underneath the integral sign, which gives a $-t$.

$$\frac{dF(s)}{ds} = -\int_0^\infty t e^{-st} g(t)\, dt$$

Moving the negative sign and moving the $t$ we get

$$-\frac{dF(s)}{ds} = \int_0^\infty e^{-st} t g(t)\, dt.$$

Which we now observe is

$$-\frac{dF(s)}{ds} = \mathscr{L}\{tg(t)\}.$$

Which is the Laplace Transform we sought.

$$\mathscr{L}\{tg(t)\} = -\frac{dF(s)}{ds} = -\frac{d(\mathscr{L}\{g(t)\})}{ds}$$

Let us consider $f(t) = g'(t)$.

$$\mathscr{L}\{g'(t)\} = \int_0^\infty e^{-st} g'(t)\, dt$$

Using integration by parts we obtain

$$\int_0^\infty e^{-st} g'(t)\, dt = g(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} g(t)\, dt.$$

$$\int_0^\infty e^{-st} g'(t)\, dt = g(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} g(t)\, dt = -g(0) + sF(s)$$

So,

$$\mathscr{L}\{g'(t)\} = sF(s) - g(0).$$

Let us find the Laplace Transform of $f(t) = g''(t)$.

$$\mathscr{L}\{g''(t)\} = \int_0^\infty e^{-st} g''(t)\, dt$$

Using integration by parts we obtain

$$\int_0^\infty e^{-st} g''(t)\, dt = g'(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} g'(t)\, dt.$$

From the previous transform we already know what the second integral is.

$$g'(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}g'(t)\,dt = -g'(0) + s\Big[s(F(s) - g(0)\Big]$$

Thus

$$\mathscr{L}\{g''(t)\} = s^2F(s) - sg(0) - g'(0).$$

Finally, let us consider $f(t) = \frac{1}{\sqrt{t}}$.

$$\mathscr{L}\{\frac{1}{\sqrt{t}}\} = \int_0^\infty e^{-st}\frac{1}{\sqrt{t}}\,dt$$

By doing a $u$-substitution with $u = st$ we now obtain

$$\frac{1}{\sqrt{s}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}}\,du.$$

Let us do a $v$-substitution now with $v = \sqrt{u}$.

$$\frac{2}{\sqrt{s}}\int_0^\infty \frac{e^{-v^2}}{v}v\,dv = \frac{2}{\sqrt{s}}\int_0^\infty e^{-v^2}dv$$

Now notice this is a Gaussian integral (Discussed in Appendix A.1) and the integral does not depend on $s$ anymore. So

$$\frac{2}{\sqrt{s}} * \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{s}}.$$

Let us do one more Laplace Transform with $f(t) = g(t) + h(t)$.

$$\mathscr{L}\{g(t) + h(t)\} = \int_0^\infty e^{-st}\big(g(t) + h(t)\big)\,dt$$

This simply is

$$\int_0^\infty e^{-st}g(t)\,dt + \int_0^\infty e^{-st}h(t)\,dt = \mathscr{L}\{g(t)\} + \mathscr{L}\{h(t)\}.$$

Thus, the Laplace Transform of a sum of functions is the sum of the Laplace Transforms of the individual functions.

$$\mathscr{L}\{g(t) + h(t)\} = \mathscr{L}\{g(t)\} + \mathscr{L}\{h(t)\}$$

### 2.2.1 Convolution

The last piece of information about Laplace Transforms we need is convolution. Convolution is where we are multiplying two Laplace Transforms together and we create a new function.

$$\mathscr{L}\{f(t)\} \bullet \mathscr{L}\{g(u)\} = F \bullet G = \left(\int_0^\infty e^{-st}f(t)\,dt\right) \bullet \left(\int_0^\infty e^{-su}g(u)\,du\right)$$

Combining the integrals we get

$$\int_0^\infty \int_0^\infty e^{-s(t+u)}f(t)g(u)\,dt\,du.$$

Now doing a change of variable with $v = t + u$ and $dv = 1$ we obtain:

$$\int_0^\infty \int_u^\infty e^{-sv}f(v-u)g(u)\,dv\,du = \int_0^\infty \int_0^v e^{-sv}f(v-u)g(u)\,dv\,du.$$

Observe now that we can simply place one integral into parentheses and obtain

$$\int_0^\infty e^{-sv}\left(\int_0^v f(v-u)g(u)\,du\right)dv = \mathscr{L}\left\{\int_0^v f(v-u)g(u)\,du\right\}.$$

Finally, we see that

$$\mathscr{L}\{f(t)\} \bullet \mathscr{L}\{g(u)\} = \mathscr{L}\left\{\int_0^v f(v-u)g(u)\,du\right\}.$$

### 2.2.2 Laplace Transform Table

Here is a list of Laplace Transforms I have shown and a few that I have not shown, but will use from here on. Also note the Laplace Transform is a linear transformation, as the sixth and the eleventh entry in the table guarantee this.

| $f(t)$ | $\mathscr{L}\{f(t)\} = F(s)$ |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $sin(at)$ | $\frac{a}{s^2+a^2}$ |
| $cos(at)$ | $\frac{s}{s^2+a^2}$ |
| $kg(t)$ | $kF(s)$ |
| $tg(t)$ | $-\frac{dF(s)}{ds}$ |
| $g'(t)$ | $sF(s)-g(0)$ |
| $g''(t)$ | $s^2F(s)-sg(0)-g'(0)$ |
| $\frac{1}{\sqrt{t}}$ | $\sqrt{\frac{\pi}{s}}$ |
| $g(t)+h(t)$ | $\mathscr{L}\{g(t)\}+\mathscr{L}\{h(t)\}$ |
| $e^{at}t^n$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\delta(t)$ | $1$ |
| $\mathscr{L}\{f(t)\} \bullet \mathscr{L}\{g(u)\}$ | $\mathscr{L}\left\{\int_0^v f(v-u)g(u)du\right\}$ |

### 2.2.3 Inverse Laplace Transforms

Most of the time, to invert a transformed function we can simply read the previous table backwards. Manipulating the equation $F(s)$ into the form of one of these can be tricky but is possible in many cases. However, sometimes we cannot do that and need a better way to invert our equation into the original variable. One such formula is the Mellin Inversion formula given by the line integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}ds.$$

This by no means is an easy formula and often requires using Cauchy's Residue Theorem.

## 2.3 Solving Differential Equations Using Laplace Transforms

Equations come up all over mathematics and the sciences that contain derivatives in the equation itself. These are differential equations, and often we would like to know what the original function is. We have several techniques for solving first and second order ordinary differential equations in the simplest of cases. However, once things get too complicated these techniques begin to fail us. Luckily, Laplace Transforms enable us to again solve for the original function.

**Example 1:** Solve $y'' + 4y = 4x$ with $y(0) = 0$ and $y'(0) = 5$. We can solve this example without using Laplace Transforms, but I will use both methods and compare the results. Using the much simpler method of solving linear second order equations we write the related auxiliary equation

$$r^2 + 4 = 0$$

and from this find the homogeneous solution to be

$$y_h = c_1 \cos(2x) + c_2 \sin(2x).$$

We determine the particular solution by using the method of undetermined coefficients. One can readily check that $y_p = x$ solves the differential equation. Thus the general solution is

$$y_g = c_1 \cos(2x) + c_2 \sin(2x) + x.$$

Using our initial conditions we can determine the values of the constants. Thus our solution is

$$y = \cos(2x) + 2\sin(2x) + x.$$

Let us see if we can duplicate the result using Laplace Transforms. For this we need to take the Laplace Transform of $y''$, $4y$, and $4x$.

$$\mathscr{L}\{y''\} = s^2F(s) - sf(0) - f'(0) = s^2F(s) - s - 5$$

$$\mathscr{L}\{4y\} = 4F(s)$$

$$\mathscr{L}\{4x\} = \frac{4}{s^2}$$

Thus the Laplace Transform of our entire problem becomes

$$s^2F(s) - s - 5 + 4F(s) = \frac{4}{s^2}.$$

We have reduced our differential equation to nothing but algebra. Solving for $F(s)$ yields

$$F(s) = \frac{\frac{4}{s^2}}{s^2+4} + \frac{5}{s^2+4} + \frac{s}{s^2+4}.$$

With some light manipulations we can turn this equation into

$$F(s) = \frac{1}{s^2} - \frac{1}{2}\frac{2}{s^2+4} + \frac{5}{2}\frac{2}{s^2+4} + \frac{s}{s^2+4}.$$

Now reading our Laplace Transform table backwards we see that

$$y = x - \frac{1}{2}\sin(2x) + \frac{5}{2}\sin(2x) + \cos(2x) = \cos(2x) + 2\sin(2x) + x.$$

This is precisely the solution we got not using Laplace Transforms. Let us try a harder example.

**Example 2:** Solve $xy'' + (3x-1)y' - (4x+9)y = 0$ with $y(0) = 0$. As it stands we cannot take the Laplace Transform of this with the products it has. Let us distribute the $y''$, $y'$, and $y$,

$$xy'' + 3xy' - y' - 4xy - 9y = 0.$$

Now let us the take Laplace Transform of each of these terms.

$$\mathscr{L}\{xy''\} = -2sF(s) - s^2F'(s) + 0$$

$$\mathscr{L}\{3xy'\} = -3F(s) - 3sF'(s)$$

$$\mathscr{L}\{y'\} = sF(s) - 0$$

$$\mathscr{L}\{4xy\} = -4F(s)$$

$$\mathscr{L}\{9y\} = 9F(s)$$

Thus the Laplace Transform of our entire equation becomes

$$-2sF(s) - s^2F'(s) - 3F(s) - 3sF'(s) - sF(s) + 4F'(s) - 9F(s) = 0.$$

This is a first order equation and after some algebra it is separable.

$$0 = (s^2 + 3s - 4)F'(s) + (3s + 12)F(s) = ((s+4)(s-1))F'(s) + 3(s+4)F(s)$$

$$F'(s)(s-1) = -3F(s)$$

$$\frac{dF(s)}{F(s)} = -\frac{3\,ds}{(s-1)}$$

Upon integrating we get

$$ln(F(s)) = -3ln(s-1) + c.$$

Now exponentiating both sides we get

$$F(s) = c(s-1)^{-3}.$$

Now matching this to a Laplace Transform, namely $\mathscr{L}\{e^{at}t^n\} = \frac{n!}{(s-a)^{n+1}}$, we find our solution to be

$$y = cx^2e^x.$$

**Example 3:** Solve $xy'' + (2x+3)y' + (x+3)y = e^{-x}$ with $y(0) = 0$ Again let us distribute the $y''$, $y'$, and $y$. We obtain

$$xy'' + 2xy' + 3y' + xy + 3y = e^{-x}.$$

Now again let us take the Laplace Transform of each of these terms:

$$\mathscr{L}\{xy''\} = -2sF(s) - s^2F'(s) + 0$$

$$\mathscr{L}\{2xy'\} = -2F(s) - 2sF'(s)$$

$$\mathscr{L}\{3y'\} = 3sF(s) - 0$$
$$\mathscr{L}\{xy\} = -F'(s)$$
$$\mathscr{L}\{3y\} = 3F(s)$$
$$\mathscr{L}\{e^{-x}\} = \frac{1}{s+1}.$$

Thus the overall Laplace Transform of the entire equation is

$$-2sF(s) - s^2F'(s) - 2F(s) - 2sF'(s) + 3sF(s) - F'(s) + 3F(s) = \frac{1}{s+1}$$

$$(-s^2 - 2s - 1)F'(s) + (s+1)F(s) = \frac{1}{s+1}.$$

Next do some algebra to find a new first order linear differential equation,

$$F'(s) - \frac{1}{(s+1)}F(s) = -\frac{1}{(s+1)^3}.$$

Now use the normal method of solving a linear differential equation to find the solution of

$$F(s) = c(s+1) + \frac{1}{3(s+1)^2}.$$

Now we take inverse Laplace Transforms and by simply reading the table backwards, we get

$$y(x) = c\delta(x) + \frac{xe^{-x}}{3}.$$

Finally, the Delta function wipes out after applying initial conditions and we are left with

$$y(x) = \frac{xe^{-x}}{3}.$$

## 2.4 Tautochrone

A variation to the Brachistochrone is the Tautochrone, which is Greek for "same time". The Tautochrone will tell us the shape of the wire with the property that no matter where you place a hollow bead, not subject to friction, it will reach the bottom of the curve in the same amount of time. Our goal is to find a function independent of starting height such that the descent time is equal. Say the bead starts at a point $(x_0, y_0)$ and ends at a lower altitude at a point $(u, v)$. We want in general to find the descent time from a point $(x, y)$ to the origin $(0, 0)$. First, we need to find a formula, $T(v)$, for the descent time. Note that $v$ identifies the the location of the bead.

$$T(v) = \int_{v=y_0}^{v=0} dt = \int_{v=y_0}^{v=0} \frac{s}{\frac{ds}{dt}}$$

This integral adds up all the time infinitesimal pieces $dt$ for the total time. Again, we will need to think about conservation of kinetic and potential energy as we did before in the Brachistochrone. Note that the difference in height will be from its start height, $v_0$, to $y$. The velocity of the bead is $\frac{ds}{dt}$.

$$E_{Potential} = mg(v_0 - y)$$

$$E_{Kinetic} = \frac{1}{2}m\left(\frac{ds}{dt}\right)^2$$

Knowing that energy is conserved we can solve for $\frac{ds}{dt}$,

$$\frac{ds}{dt} = -\sqrt{2g(v_0 - y)},$$

and here we take the negative square root as the velocity is going down the curve. Because knowing that velocity is simply distance traveled over time we know $v = \frac{ds}{dt}$. So the descent time function becomes:

$$T(v) = \int_{v=y_0}^{v=0} dt = \int_{v=y_0}^{v=0} \frac{1}{-\sqrt{2g(v_0 - y)}} ds.$$

It would be useful to know $ds$. The function $s$ is simply the arc length, with $y$ bounds of $[0, v_0]$, given by

$$s = \int_0^{v_0} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dy.$$

So taking a derivative of $s$ with respect to $y$, we find that

$$s' = \frac{ds}{dy} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Substituting this into the descent time function, we get

$$T(v) = \int_{v=0}^{v=y_0} \frac{1}{-\sqrt{2g(v_0 - y)}} ds = \int_0^{v_0} -\frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{\sqrt{2g(v_0 - y)}} dy$$

$$= \int_0^{v_0} -\sqrt{\frac{1 + \left(\frac{dy}{dx}\right)^2}{2g(v_0 - y)}} dy.$$

And this integral is hard to solve, so let us actually backtrack to

$$T(v) = -\frac{1}{\sqrt{2g}} \int_0^{v_0} \frac{s'(y)}{\sqrt{v_0 - y}} dy = -\frac{1}{\sqrt{2g}} \int_0^{v_0} s'(y) \frac{1}{\sqrt{v_0 - y}} dy.$$

This does not look much better, however this is convolution. Remember we want the time to be constant, call it $T_0$, and let us take the Laplace Transform of this function:

$$\mathscr{L}\{T_0\} = -\frac{1}{\sqrt{2g}} \mathscr{L}\{\frac{1}{\sqrt{v_0 - y}} * s'(y)\} = -\frac{1}{\sqrt{2g}} \mathscr{L}\{\frac{1}{\sqrt{v_0 - y}}\} \bullet \mathscr{L}\{s'(y)\}$$

$$= \frac{T_0}{z} = -\frac{1}{\sqrt{2g}} \sqrt{\frac{\pi}{2z}} * \mathscr{L}\{s'(y)\}.$$

Solving this for $\mathscr{L}\{s'\}$:

$$\mathscr{L}\{s'(y)\} = -\frac{T_0}{z} \frac{\sqrt{2g * 2z}}{\sqrt{\pi}} = -\frac{T_0}{z} \frac{2\sqrt{gz}}{\sqrt{\pi}} = -\frac{T_0 2\sqrt{g}}{\sqrt{z\pi}} = -\frac{T_0 2\sqrt{g}}{\sqrt{\pi}} \frac{1}{\sqrt{z}} = \mathscr{L}\{-\frac{T_0 2\sqrt{g}}{\sqrt{\pi}} \frac{1}{\sqrt{y}}\}.$$

Now just taking inverse Laplace Transforms on the far sides and lumping all the constants in one, $c$, we obtain

$$s'(y) = c\frac{1}{\sqrt{y}}.$$

Making a substitution for $s'(y)$ we get

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = c\frac{1}{\sqrt{y}}.$$

Calling $\frac{dy}{dx} = f'(y)$, and solving for it we get

$$f'(y) = \sqrt{\frac{c}{y} - 1}.$$

This is precisely the differential equation we had for the Brachistochrone. For the details of this see the derivative in section 2.1. Thus, the solution for the Tautochrone is a cycloid as well.

## 2.5 Clepsydra

In this section, I will derive the shape of a water clock- a Clepsydra. Clepsydra is Greek and comes from the Greek words meaning "to steal" and "water." The devices were used by the Greeks, Romans, and Egyptians; these time-keeping devices were similar in nature to pots or vases, with holes in the bottom. The idea was to make the water flow out at a constant rate and note the difference in volume of water in the pot.

In mathematical terms, I would like to find the smooth function of $x$ that when revolved around the $y$-axis, $\frac{dy}{dt}$ will be constant, that is, the change in height of the water. Then a hole is placed at the origin for the water to drain out of. Recall that the volume of a function, with height $y$, revolved around the $y$-axis is:

$$V(h) = \pi \int_0^y [f^{-1}(y)]^2 dy$$

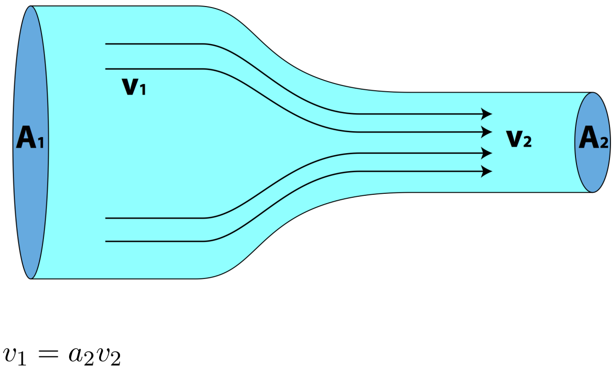
Let us take the derivative of $V$ with respect to $y$. We simply need to use the Fundamental Theorem of Calculus.

$$\frac{dV}{dy} = \pi[f^{-1}(y)]^2$$

This expression alone is not quite enough to continue, let us think about the chain rule.

$$\frac{dV}{dy}\frac{dy}{dt} = \frac{dV}{dt}$$

We know two of these terms, we just have to figure out $\frac{dV}{dt}$. In fluid dynamics, we have that the amount of fluid flowing at a velocity in a cross sectional area is equal to the amount of fluid flowing in a different sized cross sectional area at a different velocity.




$$a_1 v_1 = a_2 v_2$$

If we take this picture and rotate it 90 degrees clockwise, we would find that the volume of outflow (at the bottom) would be governed by Torricelli's Law, with $A$ as the ratio of areas and $h$ being the height of the pipe.

$$v = A\sqrt{2gh}$$

This is precisely $\frac{dy}{dt}$. We now simply need to plug in our height, $y$. Going back to our chain rule expression and plugging in what we know:

$$A\sqrt{2gyc} = \pi[f^{-1}(y)]^2.$$

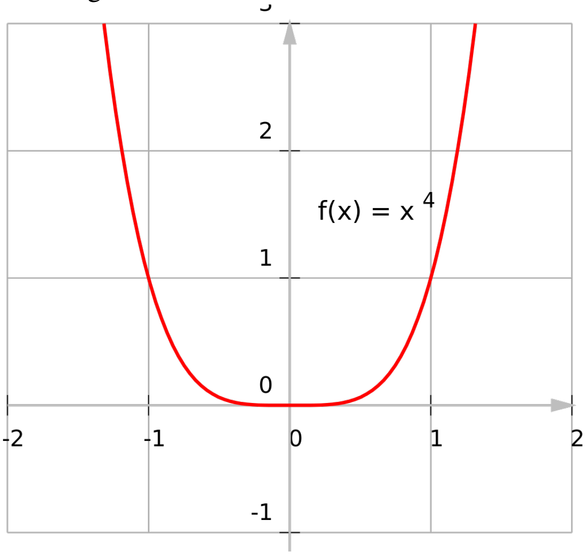
Solving this expression for $f^{-1}(y)$ and combining all constants into one, $C$, yields:

$$f^{-1}(y) = \frac{(2gy)^{\frac{1}{4}}}{\sqrt{c\pi}} = Cy^{\frac{1}{4}}.$$

Finally, taking the inverse of the inverse we find the function to be:

$$f(x) = Cx^4.$$

This is a 4th degree monomial which looks similar to the figure below.




This does look like the shape of a pot, if you can imagine this being revolved around the y-axis.

## 2.6 Calculus of Variations

### 2.6.1 The Method

In calculus we often try to minimize or maximize functions. With Calculus of Variations the goal is similar, we are trying to find the minimum or maximum of a functional. A functional is a function with the inputs as functions and outputs as functions, as opposed to what typically happens which is a function taking in real numbers and getting out real numbers. I will specifically use Riemann's, Euler's, and Lagrange's ideas. First, consider a function of 3 variables $f(x, y, y')$ with $y$ as a function of $x$. Let us take our functional as

$$M(y(x)) = \int_{x_1}^{x_2} f(x, y, y')\, dx.$$

We would like to minimize or maximize this functional. Let us assume there is some answer $y_0(x)$. Thus, we can conclude that any other function is not the solution, and let $\zeta(x)$ be any other function with $\zeta(x_1) = 0 = \zeta(x_2)$. Consider $y(x) = y_0(x) + \varepsilon\zeta(x)$. The minimum or maximum of $M(y(x))$ occurs when $y(x) = y_0(x)$, that is when $\varepsilon = 0$. It follows that our new "calculus" problem is to find the derivative of $M(y(x))$ with respect to $\varepsilon$.

$$0 = \frac{d}{d\varepsilon}M(y(x)) = \frac{d}{d\varepsilon}\int_{x_1}^{x_2} f(x, y, y')\, dx$$

Let us now differentiate underneath the integral sign, so the total derivative will become a partial.

$$\int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} f(x, y, y')\, dx.$$

Let us substitute $y(x) = y_0(x) + \varepsilon\zeta(x)$ now.

$$\int_{x_1}^{x_2} \frac{\partial}{\partial \varepsilon} f\Big(x, y_0(x) + \varepsilon\zeta(x), (y_0'(x) + \varepsilon\zeta'(x))\Big)\, dx$$

Let us take the derivative of $f$ now.

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial x}\frac{dx}{d\varepsilon} + \frac{\partial f}{\partial y}\frac{dy}{d\varepsilon} + \frac{\partial f}{\partial y'}\frac{dy'}{d\varepsilon}\, dx$$

Upon taking partials of $f$ and with a change in notation, this becomes:

$$\int_{x_1}^{x_2} 0 + f_y\zeta(x) + f_{y'}\zeta'(x)\, dx = \int_{x_1}^{x_2} f_y\zeta(x) + f_{y'}\zeta'(x)\, dx.$$

Now let us split the integral into two separate ones and use integration by parts on the second one, let $u = f_{y'}$ and $v' = \zeta'(x)$.

$$\int_{x_1}^{x_2} f_y\zeta(x)\, dx + \int_{x_1}^{x_2} f_{y'}\zeta'(x)\, dx = \int_{x_1}^{x_2} f_y\zeta(x)\, dx + f_y\zeta(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \zeta(x)\frac{d}{dx}f_{y'}\, dx$$

Now using the stipulation that $\zeta(x_1) = 0 = \zeta(x_2)$, the middle term disappears. Now let us combine the integrals again,

$$\int_{x_1}^{x_2} f_y\zeta(x) - \zeta(x)\frac{d}{dx}f_{y'}\, dx = \int_{x_1}^{x_2} \zeta(x)\Big(f_y - \frac{d}{dx}f_{y'}\Big)\, dx = 0$$

Note that $\zeta(x)$ is any function, thus $\zeta(x)$ or $(f_y - \frac{d}{dx}f_{y'})$ should be zero for the integral to be zero. Taking $\zeta(x)$ to be nontrivial we conclude that $(f_y - \frac{d}{dx}f_{y'})$ should be zero.

$$0 = f_y - \frac{d}{dx}f_{y'}$$

This famous result is known as the Euler-Lagrange Equation, or in a different notation:

$$\boxed{\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.}$$

Therefore solving this differential equation will give us the function that minimizes our functional.

### 2.6.2 Beltrami's Identity

In this section, I will derive a useful corollary of the Euler-Lagrange Equation. Beltrami's Identity is a simplification that can be made to the Euler-Lagrange equation if $f$ does not depend on $x$. From the Euler-Lagrange equation we know that

$$\frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\partial f}{\partial y'}.$$

If we take the total derivative of $f(x, y, y')$ with respect to $x$ we get:

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx}.$$

But we know that $\frac{\partial f}{\partial x} = 0$, as $f$ does not depend on $x$. Thus we obtain the following,

$$\frac{df}{dx} - \frac{\partial f}{\partial y} y' - \frac{\partial f}{\partial y'} y'' = 0.$$

Plugging the Euler-Lagrange equation into the second term, we get

$$\frac{df}{dx} - y' \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y'} y'' = 0.$$

Now observe that the last two terms can be rewritten using the product rule of derivatives as:

$$\frac{df}{dx} - \frac{d}{dx}\Big(\frac{\partial f}{\partial y'} y'\Big) = \frac{d}{dx}\Big(f - \frac{\partial f}{\partial y'} y'\Big) = 0.$$

Upon integrating, we are left with Beltrami's Identity:

$$\boxed{f - y' \frac{\partial f}{\partial y'} = C.}$$

## 2.7 Brachistochrone Revisited

We will now revisit the Brachistochrone using the Calculus of Variations. Our goal is as it was before: we need to minimize the time of descent. Recall that $t = d/v$, so we need to compose a functional in this form.

$$t(y(x)) = \int \frac{d}{v}\, dx$$

Where $d$ is the arc length of the function in question, and $v = \sqrt{2gy}$ as above, so our functional becomes:

$$t(y(x)) = \int \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}\, dx.$$

As $\sqrt{2g}$ is just a constant, we can eliminate it and find the function that we need to minimze is:

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{y}}.$$

As $f$ does not depend on $x$ we can utilize Beltrami's Identity:

$$f - y' \frac{\partial f}{\partial y'} = C.$$

Upon taking a partial of $f$ we are left with:

$$\frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'(x)^2}{\sqrt{y}\sqrt{1 + y'^2}} = C.$$

Finding a common denominator and simplifying, we are left with the differential equation:

$$\sqrt{\frac{C}{y} - 1} = y'.$$

This is precisely the differential equation we had found before with the solution being a cycloid.

*Chapter 3*

# History of Time

In this section, I discuss how these math problems are relevant to history and culture and how they are a part of the human experience. I first talk about the connection between the Tautochrone and clock making. After, I have discussed how we started to obtain and worry about accurate clocks, I discuss longitude and its importance. Still moving forward, as clocks become more and more accurate we can begin to worry about unifying time.

## 3.1 Tautochrone to Clocks

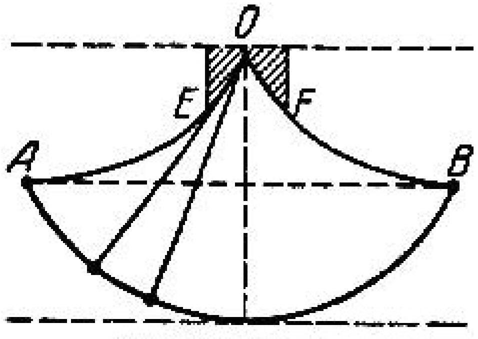
In this section we take a small break from heavy mathematics and discuss the importance of clocks and chronometers. Being able to tell time is clearly important in order to make sure you are "on time" for whatever you might need to go to, e.g. work, class, meeting, etc. In the past, it was important to keep track of the time of day to make sure all of your important tasks were completed in a timely fashion e.g. farming, maintaining livestock, and other chores relevant at the time. Keeping track of the days were also important, days such as the Sabbath, the holy day, as this was a day of rest. We first examine the earliest form of time-keeping and explore how it evolved through the ages.

For many millennia humans have kept track of time, as early as roughly 1500 BCE the Ancient Egyptians, Babylonians, and Greeks in have made time keeping devices. The Ancient Egyptians were the first to use sundials. They built an obelisk- a tall stone pillar, which the sun moving throughout the day would cause the shadow of the obelisk to move. The movement of the shadow of the clock indicated a time. This was a measurement of local time, as the shadow was relative to where the sun was in the sky, and thus depended on the location of the obelisk on Earth. We see that the older civilizations utilized celestial objects to track time.

A more accurate time keeping device also came from the Egyptians in approximately 1400 BCE. The clepsydra was used by the Egyptians, Greeks, and Romans. The word clepsydra comes from the Greek language "cleps" which means to steal and "-ydra" meaning water. The clepsydra is a device in which water escaped from, in a precise manner- the water "stole away." The clepsydra was used to regulate speech lengths in the Roman senate. The phrase that time has run out might come from this, referring to the water in the clepsydra; the water has ran out and so has the time. Originally, the day was split into 12 equal parts and the night was split into 12 equal parts; since, we have split the day into 24 equal parts as the seasons do not allow a night and day part to truly be equal.

The clepsydrae were typically clay pots with one or several holes in the bottom or sides which allowed water to escape at a certain rate. Most importantly, the rate at which water flowed out was not uniform. As the volume of water decreased the rate at which the water also decreased- this is Torricelli's Law. Essentially, the water flows out of the pot slower as the pot becomes more empty. We see that the Egyptians overcame this obstacle by having slanted sides, similar to $y = ax^4$ (as derived in section 2.5). Another way they overcame this obstacle was to keep the volume of water constant, which caused the rate at which water escaped to be constant. They then tracked water in a normal right cylinder to track time; this was called an outflow clepsydra. Clepsydra were often calibrated to sundials in order to maintain their regularity.

Of course, there are many other forms of time keeping devices. Candle clocks and incense clocks function similar to clepsyrdae, that is, the rate at which the material burns is constant. We also have the hourglass, in which sand (or another material) flows from the top to the bottom of a container, measuring a fixed amount of time. Now we will examine one of the most important inventions in all of history- the mechanical clock. Early mechanical clock were described using the Greek word "horologia" which comes from the Greek roots "horo" meaning hour and "logia" meaning to tell. Galileo was the first to attempt to make a pendulum clock, but he never truly perfected it and credit goes to Dutch mathematician and scientist Christiaan Huygens in 1658 for the one of the first pendulum clocks. The key was to make the pendulum isochronous; that is, the pendulum would always swing in a consistent pattern. Simple pendulums are only isochronous under small amplitudes. which essentially makes it impossible to accurately keep track of time aboard a moving ship. Huygens knew he could put some curve above the pendulum to control how it would swing, but he did not know what type of curve to put. Huygens was an early mathematician and he also attempted to tackle the Brachistochrone problem, eventually becoming a master of the cycloid. He decided to check if a cycloid was the correct curve to make an isochronous pendulum, and it was. Thus the cycloid, which is the tautochrone, is the correct curve that would make a pendulum isochronous as pictured below.



Cycloid pendulum

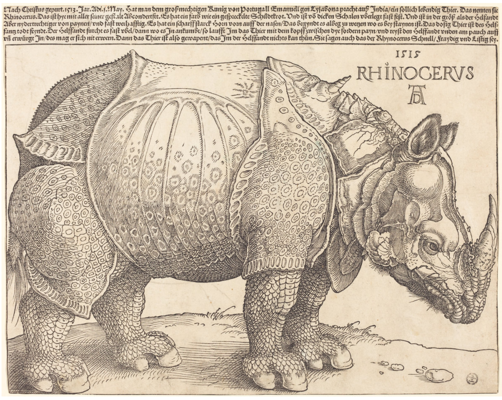
More precisely, Huygens had to find the evolute of the cycloid, which is the locus of centers of curvature of a curve's normal lines. The wonderful thing about the evolute of a cycloid is that it is a cycloid that has been shifted out of phase. Thus, we see how the tautochrone has its place in clock design.

## 3.2 Clocks to Longitude

We now have developed an early mechanical clock. The isochronous, or tautochrone, pendulum allowed for large amplitudes that in theory allowed for the clock to maintain precision even on the rough seas. Sadly, this clock only works in theory. Unfortunately, although Huygens spent a few more decades trying to perfect his pendulum clock, he was never able to and his original clock was nowhere close to being able to maintain accuracy aboard a ship. However, his work on clocks and evolutes can still be seen in many different areas of math and science.

The problem of determining longitude was a daunting issue for maritime and astronomical sciences. Latitude was an easy calculation to make because one could easily measure the altitude (the height above the horizon) of the Sun at noon or a particularly bright and prominent star, such as Polaris the North Star. The altitude of the star or Sun was used to determine latitude. However, determining longitude was much harder and made oceanic journeys quite risky.

In 1515 a rhinoceros had just arrived safely, by boat, in Lisbon, Portugal from India. This was a monumental feat as this was the first rhinoceros in Europe during human times. Later in 1515, King Manuel I of Portugal had sent the rhinoceros to Pope Leo X, who had received an elephant from Manuel I the year prior. King Manuel had sent the rhinoceros to bribe the Pope to gain his favor over Spain. Pope Leo X was a lover of exotic animals. However, due to a sudden storm and navigation problems the ship had crashed and the rhinoceros died with it. Thus we see an early accident at sea caused by trouble with navigation. Luckily in that same year, a German painter and print-maker Albrecht Dürer made a woodcut of that rhinoceros, only having a written description of the rhino and a sketch by an unknown artist- so at least that rhinoceros is immortal in a way. Pictured below is an image of the woodcut. This story accounts for a part of the title of this project "The Pope’s Rhinoceros."

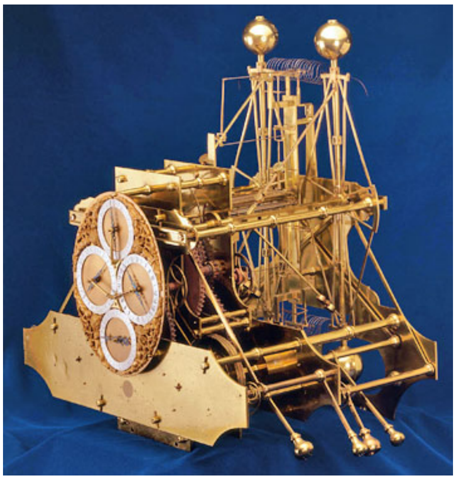



The Scilly Naval disaster of 1707 is considered one of the worst maritime disasters of the British navy. Due to bad weather the fleet was pushed off course and killed by crashing into rocks. Over 1500 sailors died due to bad weather, but more relevantly the navigators were unable to determine longitude successfully and did not know their actual location. If the navigator had a better knowledge of their location they could have avoided the rocks. The problem of finding accurate longitude became so notable that in 1714 The Longitude Act was passed. It established the Board of Longitude and offered a reward for anyone that could find a simple and practical method to "precisely" determine a ship’s longitude.

The earliest and least accurate method to travel at sea was originally by dead reckoning. Dead reckoning is the use of your current direction and speed to calculate the distance you would travel in a certain time. This navigation process was prone to compounding error and as a result was disastrous. Eventually, scientists and inventors tried to come up with more accurate ways to determine longitude. The two main methods to determine longitude were either utilizing the eclipses of Jupiter’s moon or comparing a local time to a reference time and using that difference to calculate your longitude. Earth revolves 360° in 24 hours, so in 1 hour the Earth revolves 15°. One could determine the longitude by comparing local time with a reference clock, such as Greenwich in England, and every hour of difference corresponded to

15° of longitude.

While using the eclipses of Jupiter's moons on land was quite easy, on a ship it was a nearly impossible task to accurately time the eclipses and then to accurately determine longitude. The astronomers fought long and hard with clock designers for the Longitude Act reward, but the clock designers eventually were able to design a precise marine chronometer. John Harrison in 1735 built his first chronometer, and over the next several decades perfected it and fought off astronomers from winning any of the Longitude Act reward. His final clock was able to fit in a captain's pocket and only lost about 5 seconds in about two months. Pictured below are Harrison's first chronometer, H1, on the left and last chronometer he made, H4, on the right.



## 3.3 Longitude to Time Unification

While Harrison's watches were magnificent, clocks and chronometers still continued to evolve. As the precision continued to increase, we became worried more about unification of those clocks. Albert Einstein and Henri Poincaré were the first to tackle the problem of time unification. Einstein was a German physicist and Poincaré was a French engineer, mathematician, and physicist. In the late 1800's clock synchronization was trying to be implemented, the goal was to eliminate discrepancies between different cities. At the start of the 19th century many clocks did not even have minute hands, there simply was not enough precision necessitate with those yet. This time synchronicity became essential to have an effective railroad schedule. As covered in the last section, accurate clock synchronization helped determine accurate longitude. In Paris, pneumatic tubes were used in attempt to unify time across the city by using pressurized air. By the late 19th century telegraphs were sending signals over the Atlantic ocean to establish this time synchronization. These trans-Atlantic cables were quite a feat and enabled accurate longitude measurements.

In order to achieve this time synchronization, the Einstein-Poincaré synchronization procedure was used. We first define simultaneous events, if for instance a train arrives at a station at 7pm, that means the the arrival of the train and a clock pointing at 7pm happened at the same time-they were simultaneous events. To synchronize two clocks, clock A and clock B, we send a light pulse from A to B and B will reflect this light pulse immediately back towards clock A. These two clocks are synchronized if the time between clock A and B is the same as the time between clock B and clock A. Furthermore, we know that if clocks A and B are synchronized and clocks B and C are synchronized, then clocks A and C are synchronized.

With this time synchronization, we designed the time zones to have a standard in which to compare time across Earth. Greenwich was selected as the "origin" because the Royal Observatory in Greenwich was one of the earliest places to have an incredibly accurate clock. Greenwich was selected with much disdain to Paris's taste, as the Paris observatory was also a powerhouse with an accurate clock. After much political debate Greenwich was chosen to be the Prime Meridian. However, Paris got to hold the official meter and kilogram in a vault. This advance in the study of time and clocks enabled Einstein to make huge leaps in the world of physics.

*Chapter 4*

# Quantum Mechanics

## 4.1 Modern Physics

We now venture into the world of modern physics; modern physics is where classical mechanics stops being able to predict things. Modern physics has essentially had its place post 1900's and some discoveries starting to be made just before 1900. In the modern physics realm things are either on an extremely small scale or an extremely large scale. Einstein, with his study of clock synchronicity, was a father of modern time and an essential start to the world of modern physics. He is known best for his energy-mass relationship, $E = mc^2$. However, Einstein has two fundamental postulates. First is his principle of relativity, which states that the laws of physics are the same in every inertial frame of reference. Until the turn of the 20th century, many scientists thought that light had to travel through a medium-aether (sometimes ether). Michelson and Morley had experimental evidence to support that aether did not exist and from this Einstein came up with his second postulate. His second postulate states that: the speed of light in a vacuum is the same in all inertial frames of reference and is independent of the motion of the source. This postulates entails the speed limit of the universe, nothing can go faster than light. The speed of light, $c$, in a vacuum is 299,792,458 m/s. Einstein was able to devise his theory of Special Relativity from his study of clocks.

Special Relativity was a fundamental shift in how we thought about physics. When objects start moving at relativistic velocities, which are large fractions the speed of light, classical mechanics begin to fail us. One consequence of special relativity is that velocity addition is now different, as objects can never be moving faster than $c$. For example, we have a train moving at 200 miles per hour and someone on the train throws a ball at a speed of 50 miles per hour. For someone not on the train to obtain the velocity of the ball as they would see it, they would simply add the velocity of the train and the ball and get 250 miles per hour. However, as soon as the speeds get relativistic, this normal addition of velocities fail us. For example, we now have a rocket traveling at 30,000 kilometers per second, $0.1c$, and someone shining a laser on that rocket, with a velocity, $c$. By simply adding the velocities we would get $1.1c$, which violates the universal speed limit- something failed us. We need to use a relativistic velocity addition formula,

$$v_{net} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}.$$

So in our example, we find that

$$v_{net} = \frac{0.1c + c}{1 + \frac{0.1c * c}{c^2}} = \frac{1.1c}{1 + 0.1} = c.$$

Thus, the universal speed limit was not broken after all. We can play around with the numbers and see that no matter what is selected the universal speed limit will never be broken.

Another major consequence of Special Relativity is that of time dilation and length contraction. Objects seen in a moving frame of reference will experience length contraction, that is they will appear shorter in the direction of motion. Clocks (and to be specific all objects) in a moving frame of reference will also experience time dilation; that is time will appear as longer in the moving frame. A Dutch physicist, Hendrik Lorentz, had established the Lorentz Transformations before Einstein came along with his Special Theory of Relativity and Lorentz had devised how an object behaves in a moving reference frame. So there is a Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

that will appear in the following equations mathematically describing length contraction and time dilation. For time dilation, the time in the moving frame $t_m$ with velocity $v$ and the time in a resting frame $t_0$ will be given by the following equation:

$$t_m = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma t_0.$$

Similarly, we have the length contraction formula for resting length $L_0$ and moving length $L_m$:

$$L_m = L_0\sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}.$$

Other than these consequences of Special Relativity, there are relativistic mass issues and Einstein's famous mass-energy relationship. Einstein states that $ds$, the space-time separation of any two events, is invariant; that is, they are the same regardless of reference frame. We can consider a "four-dimensional" case of the Pythagorean Theorem, $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$ is invariant. To derive Einstein's famous equation, we will use two 1-D reference frames $x$,$t$ and $x'$,$\tau$. As $ds$ must be the same in either reference frame, we find that:

$$dx^2 - dt^2 = ds^s = dx'^2 - d\tau^2.$$

For convenience we can convert $dt$ to meters so,

$$dx^2 - (c\,dt)^2 = dx'^2 - (cd\tau)^2.$$

In $x'$'s reference frame, $dx' = 0$, so we can simplify to obtain:

$$c^2 d\tau^2 = c^2\,dt^2 - dx^2.$$

Then, upon dividing by $dt^2$ and noticing that $\frac{dx}{dt} = v$ we get,

$$c^2\left(\frac{d\tau}{dt}\right)^2 = c^2 - \left(\frac{dx}{dt}\right)^2 = c^2 - v^2.$$

Now observe that $\gamma$, the Lorentz factor, is $\frac{dt}{d\tau}$ and taking a binomial expansion of $\gamma^{-1}$ that we can keep just two terms of the expansion, as we can assume velocity is much smaller than $c$, hence,

$$\frac{d\tau}{dt} = \gamma^{-1} = 1 + \frac{1}{2}\frac{v^2}{c^2}.$$

Upon substituting, multiplying our $ds^2$ equation by $m^2$, and some algebra we find:

$$m^2c^2\gamma^{-2} = m^2c^2 - m^2v^2$$

$$m^2c^2 = m^2c^2\gamma^2 - m^2v^2\gamma^2.$$

Now grouping and using the fact that relativistic momentum is $p = \gamma mv$, we find that:

$$m^2c^2 = (mc\gamma)^2 - (mv\gamma)^2 = (mc\gamma)^2 - p^2.$$

Now using the binomial expansion of $\gamma$ we obtain:

$$m^2c^2 = \Big[mc\big(1 + \frac{1}{2}\frac{v^2}{c^2}\big)\Big]^2 - p^2$$

$$m^2c^2 = \Big[mc + \frac{1}{2}\frac{mcv^2}{c^2}\Big]^2 - p^2$$

$$m^2c^2 = \Big[\frac{1}{c}\big(mc^2 + \frac{1}{2}mv^2\big)\Big]^2 - p^2.$$

Now notice that $mc^2$ is the rest energy of a particle and $\frac{1}{2}mv^2$ is the kinetic energy of a particle. The sum of a particle's rest energy and kinetic energy is simply the total energy of a particle $\varepsilon$, so we then find:

$$m^2c^2 = \Big[\frac{1}{c}\varepsilon\Big]^2 - p^2$$

$$m^2c^2 = \frac{1}{c^2}\varepsilon^2 - p2$$

$$m^2c^4 = \varepsilon^2 - p^2c^2$$

Upon rearranging and grouping, we find Einstein's total relativistic energy equation.

$$\boxed{\varepsilon^2 = (mc^2)^2 + (pc)^2}$$

This equation is fundamental and has lead to both a better understanding of how the sun produces energy and the creation of atomic bombs. Notice when the particle has no momentum we find that

$$\varepsilon = mc^2.$$

This is the equation popularly associated with physics and Einstein.

In 1900, German physicist Max Planck tried to explain blackbody radiation (radiation from warm objects) using classical mechanics and could not. He could only produce experimentally known blackbody radiation spectrum by making the assumption that the energy of these waves was quantized, that is the energy came in discrete multiplies. Planck was confused with his findings that the energy of the wave, E, the frequency, f, were directly proportional by integer, n=1,2,3..., multiplies of his constant $h = 6.6 * 10^{-34} J\,s$.

$$\boxed{E = nhf}$$

In 1905, Einstein, who was curious about Planck's work, found that the the photon is a quantum packet of electromagnetic radiation. He also found that

$$\boxed{E = hf = \hbar\omega},$$

where $\hbar$ is Planck's reduced constant $1.05 * 10^{-34} Js$ and $\omega$ is the angular frequency.

In 1922, Louis de Broglie, a French physicist, proposed the wave-particle duality. He found through experiment that waves sometimes act like particles, and according to Special Relativity, light (the photons) have energy $E = pc$. De Broglie makes the argument that

$$E = hf = h\frac{c}{\lambda},$$

an equation for waves carries over to particles as well hence,

$$\boxed{p = \frac{h}{\lambda}}.$$

With this, I conclude this introduction into modern physics and I can proceed into the next section. We see that many major discoveries started happening at the turn of the 20th century, this was in part due to major technological advancements and our study of time.

## 4.2 Schrödinger Equation

In many Quantum Mechanics textbooks the Schödinger Equation is typically just given, as if it is from the Bible or mankind has always had it. This is not the case, and it can be derived quite simply in fact. We need to know some modern physics and the wave equation. De Broglie suggested not only that light has particle characteristics, but that classical particles, such as protons and electrons, have wave properties. Thus we get into the duality of wave and particles; particles can behave like waves and waves can behave like particles. De Broglie associated a particle's momentum, $p$, with its wavelength and Planck's constant, $h$, $p = \frac{h}{\lambda}$. We also need to get Einstein's relativistic total energy given by $\varepsilon^2 = p^2c^2 + m^2c^4$ and that $\varepsilon = hf = \hbar\omega$ where $f$ is the frequency, $\omega$ is the angular frequency, and $\hbar$ is Planck's Reduced constant i.e. $\frac{h}{2\pi}$. Finally, we need to obtain the wave equation governing electromagnetic waves in free space. This wave equation is derived from Maxwell's Equations in free space.

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0$$

$$\nabla \cdot \vec{B} = 0$$

We start by using the first equation and applying the del operator to both sides:

$$\nabla \times (\nabla \times \vec{E}) = \nabla \times -\frac{\partial \vec{B}}{\partial t}.$$

On the left hand side of this equation we use the true triple vector product order for non-commutative vectors and $\nabla \times \vec{E} = 0$.

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E} = 0 - \nabla^2\vec{E} = -\nabla^2\vec{E}$$

Now on the right hand side of the original equation, we can take the derivative and constant out of the cross product. Next, we can substitute Maxwell's second equation in and take the derivative.

$$\nabla \times -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{B}) = -\frac{\partial}{\partial t}\frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = -\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Now, by equating the left and the right side and multiplying by $-1$ we obtain the electromagnetic wave equation in three-dimensions.

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

In one-dimension this becomes:

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

One can easily verify that $E(x,t) = E_0 e^{i(kx-\omega t)}$ is a solution.

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E_0}e^{i(kx-\omega t)} = 0$$

$$\left(-k^2 + \frac{\omega^2}{c^2}\right)\vec{E_0}e^{i(kx-\omega t)} = 0$$

Upon taking a non-trivial amplitude for the wave we find the dispersion relation of light in free space:

$$k = \pm\frac{\omega}{c}.$$

Now recall that Einstein says that $\varepsilon = \hbar\omega$ and De Broglie says that $p = \hbar k$. We can rewrite our solution as:

$$E(x,t) = E_0 e^{\frac{i}{\hbar}(px-\varepsilon t)}.$$

Again, we can verify this is a solution to the wave equation and find that:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E_0}e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

$$-\frac{1}{\hbar^2}\left(p^2 - \frac{\varepsilon}{c^2}\right)\vec{E_0}e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0.$$

Taking the non-trivial solution again we can find the massless relativistic energy $\varepsilon^2 = p^2c^2$. This is comforting as electromagnetic waves, light, is massless. Let us now reverse engineer this solution for a particle with mass. We now go to relativistic energy with mass $\varepsilon^2 = p^2c^2 + m^2c^4$. Since we are no longer dealing with electromagnetic waves, let us now call our function a *wave* function, $\Psi$, that is a function that came from a wave equation.

$$-\frac{1}{\hbar^2}\left(p^2 - \frac{\varepsilon}{c^2} + m^2c^2\right)\Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

Still reverse engineering this equation we get:

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right)\Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0$$

With an analogy to Electromagnetics, recall that number of protons, $S$, where $\mu_0$ is the permeability of free space constant is given by:

$$S = \frac{1}{c\mu_0}E^2.$$

We demand that our wave function be normalizable, that is the particle must be found somewhere.

$$\int_{-\infty}^{\infty} \Psi^*\Psi \, dx = 1$$

Now simply using this normalization condition, we obtain the Klein-Gordon equation for a free particle in one-dimension, which is a relativistic equation.

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$$

Now going back into three-dimensions is easy, the partial simply becomes a Laplacian.

$$\nabla^2\Psi - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$$

This is the Klein-Gordon equation in three-dimensions, which will describe any spinless particle, such as the Higgs Boson. The Klein-Gordon Equation describes orbital momentum. From here, let us make a non-relativistic equation. Using Einstein's energy relation, a Taylor series expansion, and recalling that kinetic energy is $T = \frac{p^2}{2m}$:

$$\begin{aligned}\varepsilon &= mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \\ &\approx mc^2\left(1 + \frac{p^2}{2m^2c^2}\right) = mc^2 + \frac{p^2}{2m} = mc^2 + T.\end{aligned}$$

Using this approximation into our wave function we find:

$$\Psi = \Psi_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = \Psi_0 e^{\frac{i}{\hbar}(px-mc^2t-Tt)} = \Psi_0 e^{-\frac{i}{\hbar}mc^2t} e^{\frac{i}{\hbar}(px-Tt)}$$

Now because we are assuming non-relativistic velocities, i.e. $mv \ll mc$ and $p^2 \ll m^2c^2$ observe that the first exponential oscillates quickly and the second one slowly, let us call the slow one $\phi$. Thus, $\Psi = \Psi_0 e^{-\frac{i}{\hbar}mc^2t}\phi$. Let us take the second partial with respect to $t$ and then substitute this into the one-dimensional Klein-Gordon Equation.

$$\frac{\partial\Psi}{\partial t} = -\frac{i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\phi + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}$$

$$\begin{aligned}\frac{\partial^2\Psi}{\partial t^2} &= \left( -\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right) + \left( -\frac{i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t} + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial^2\phi}{\partial t^2}\right) \\ &= \left( -\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{2i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right) + e^{-\frac{i}{\hbar}mc^2t}\frac{\partial^2\phi}{\partial t^2}\end{aligned}$$

The term in the parentheses is very large and the second term is very small, so neglecting the second term and using these partials in the Klein-Gordon equation we obtain

$$\frac{\partial^2\Psi}{\partial x^2} - \frac{m^2c^2}{\hbar^2}\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{1}{c^2}\left( -\frac{m^2c^4}{\hbar^2}e^{-\frac{i}{\hbar}mc^2t}\phi - \frac{2i}{\hbar}mc^2 e^{-\frac{i}{\hbar}mc^2t}\frac{\partial\phi}{\partial t}\right).$$

Putting $\Psi$ into this and rearranging we obtain:

$$e^{-\frac{i}{\hbar}mc^2t}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{2im}{\hbar}\frac{\partial\phi}{\partial t}\right) = 0.$$

The exponential will never be equal to 0, so divide by it and then rearranging we obtain:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{2im}{\hbar}\frac{\partial\phi}{\partial t} = 0$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} = i\hbar\frac{\partial\phi}{\partial t}.$$

This is the one-dimensional Schödinger Equation for a free particle; that is, it has zero potential energy. We can again generalize to three-dimensions by replacing the 2nd partial of $\phi$ with respect to x with a Laplacian. Notice, though that $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is the total energy of a particle, and as it has no potential it is pure kinetic energy. If we instead make this the total energy for a nonzero potential we get $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)$. Putting this into the previous equation:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2} + V(x,t)\phi = i\hbar\frac{\partial\phi}{\partial t}.$$

Generalizing this to three-dimensions we obtain the time-dependent Schödinger Equation:

$$\frac{-\hbar^2}{2m}\nabla^2\phi + V(x,y,z,t)\phi = i\hbar\frac{\partial\phi}{\partial t}.$$

For historical reasons $\phi$ has been replaced with $\Psi$, so many people are used to seeing the time-dependent Schödinger Equation as:

$$\boxed{\frac{-\hbar^2}{2m}\nabla^2\Psi + V(x,y,z,t)\Psi = i\hbar\frac{\partial\Psi}{\partial t}.}$$

Finally, I end with giving the one-dimensional time-independent Schrödinger Equation, as this is the equation to solve to give us the basis for the quantum mechanical system we are dealing with.

$$\boxed{\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\Psi = E\psi.}$$

## 4.3 Hilbert Space

We now venture into the playground of Quantum Mechanics - Hilbert Space. Hilbert spaces are an abstract linear vector space that is also an inner product space and a complete metric space. First, I will briefly discuss Dirac Notation, which allows ease in dealing with the formalism associated with Quantum Mechanics. It is compromised of bras, kets, and bra-kets. Kets are column vectors which are elements of the vector space. A ket or ket vector $\alpha$ is denoted by $|\alpha\rangle$. Bras are row vectors which belong to the dual space associated with our space. A bra $\beta$ or vector bra is denoted $\langle\beta|$. Dirac denoted the inner product by the bra-ket $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle$. Finally, observe that for every ket there exists a unique bra and for every bra there exists a unique ket. Bra and ket are related in the following way: $|a\alpha\rangle = a\,|\alpha\rangle = a^*\,\langle\alpha| = \langle a\alpha|$.

As Hilbert Space is a vector space, I should state the properties of a vector space. A set of vectors $|\alpha\rangle$,$|\beta\rangle$, and $|\gamma\rangle$ with scalars a,b,c are called a linear vector space if the following axioms hold:

1. Vector Addition

A. $|\alpha\rangle + |\beta\rangle = |\gamma\rangle$; which still belongs to the vector space, that is the vector space is closed under addition.

B. $\big(\,|\alpha\rangle + |\beta\rangle\,\big) + |\gamma\rangle = |\alpha\rangle + \big(\,|\beta\rangle + |\gamma\rangle\,\big)$; that is, the vector space is associative.

C. There exists a $|0\rangle$ such that $|\alpha\rangle + |0\rangle = |\alpha\rangle$; that is, there is an additive identity to each ket.

D. There exists a $|-\alpha\rangle$ such that $|\alpha\rangle + |-\alpha\rangle = |0\rangle$; that is, there is an additive inverse to each ket.

2. Scalar Multiplication

A. $a\,|\alpha\rangle = |\gamma\rangle$ which still belongs to the vector space; that is, the vector space is closed under scalar multiplication.

B. $a\big(\,|\alpha\rangle + |\beta\rangle\,\big) = a\,|\alpha\rangle + a\,|\beta\rangle$; that is, there is a distribution of scalars.
C. $a(b\,|\alpha\rangle) = (ab)\,|\alpha\rangle$.
D. $0\,|\alpha\rangle = |0\rangle$ and $1\,|\alpha\rangle = |\alpha\rangle$.
E. $|-\alpha\rangle = -1\,|\alpha\rangle$.

Hilbert Space is also an inner product space, so I will state the properties of an inner product space. In general, the inner product $\langle\alpha|\beta\rangle$ is a complex number. Given a set of vectors $|\alpha\rangle$,$|\beta\rangle$, and $|\gamma\rangle$ with scalars $b$ and $c$ the inner product satisfies the following axioms:
A. $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$; that is, inner products are conjugate symmetric.
B. $\langle\alpha|\alpha\rangle \geq 0$ and $\langle\alpha|\alpha\rangle = 0$ if and only if $|\alpha\rangle = 0$.
C. The inner product is linear in ket, $\langle\alpha|b\beta + c\gamma\rangle = b\,\langle\alpha|\beta\rangle + c\,\langle\alpha|\gamma\rangle$.
D. The inner product is sesquilinear in bra, $\langle b\beta + c\gamma|\alpha\rangle = \big(\,\langle b\beta + c\gamma|\alpha\rangle\,\big)^* = b^*\,\langle\beta|\alpha\rangle + c^*\,\langle\gamma|\alpha\rangle$.
E. The norm of a vector is given by $\sqrt{\langle\alpha|\alpha\rangle} = ||\alpha||$ and is always a real number, this gives a sense of "length" to the vector.

Finally, Hilbert Space is a complete metric space. A space with a metric, a function $d$, that measures *distance*, is a metric space if $d$ satisfies the following for all $x$,$y$,$z$ that exist in the space:
A. $d(x, y) = d(y, x)$; that is, the distance function is symmetric. Essentially, The distance from point A to point B is the same as the distance from point B to point A.
B. $d(x, y) = 0$ if and only if $x = y$.
C. $d(x, y) + d(y, z) \geq d(x, z)$; that is, the triangle equality holds.
D. We call a metric space complete if it also satisfies the property that every Cauchy sequence converges to a limit within the space.

Do note however that a space can be complete with respect to any arbitrary norm- it is specifically the inner product metric that makes the space a Hilbert Space.

We now examine some common examples of these abstract notions. A common vector space is $\mathbb{R}^n$, this is the normal vectors of dimension $n$. We exist and think in $\mathbb{R}^3$; that is, everything has length, wi dth, and height. A common metric in $\mathbb{R}^3$ is $d = \sqrt{x^2 + y^2 + z^2}$.

## 4.4 A Laplace Transform Solution to Schrödinger Equation

An important solution to the Schrödinger Equation is one with the harmonic oscillator potential. This is a great case as this would be the first term in a Taylor expansion around a minimum. I will consider the one-dimensional case and let $V = \frac{1}{2}kx^2$. We first solve the time-independent Schrödinger Equation. This is typically done using either an algebraic "ladder operator" or using a power series solution method. I will use a rarely used method to obtain the same results- Laplace Transforms.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\Psi = E\psi$$
$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\Psi = E\psi$$

We can define some new constants and a variable to simplify the calculations, let $x = c\xi$ with $c = \sqrt[4]{\frac{\hbar^2}{mk}}$ and $\lambda = \frac{2E}{\hbar}\sqrt{\frac{m}{k}}$. We can now see that the equation reduces to:

$$\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0.$$

We can then approximate this equation as:

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2\psi.$$

This has the solution:

$$\psi = Ae^{\frac{-\xi^2}{2}} + Be^{\frac{\xi^2}{2}}.$$

As $\psi$ represents a wave function, we can use the fact that as $\xi \to \pm\infty$, the second term blows up, so $B = 0$, and we can just look at the first term and another function $h(\xi)$. So we obtain a new solution:

$$\psi = Ae^{\frac{-\xi^2}{2}}h(\xi)$$

Upon substitution of this solution back into our original reduced equation we obtain:

$$h'' - 2\xi h' + (\lambda - 1)h = 0,$$

and the usual way to continue at this point would be to use a power-series solution; I will use Laplace Transforms. I will now take the Laplace Transform of the various terms of this differential equation. Let us denote $\mathscr{L}\{h\}$ as $F(s)$.

$$\mathscr{L}\{h''\} = s^2F(s) - sh(0) - h'(0)$$

$$\mathscr{L}\{-2\xi h'\} = 2sF'(s) + 2F(s)$$

$$\mathscr{L}\{(\lambda - 1)h\} = (\lambda - 1)F(s)$$

Thus our transformed equation is:

$$s^2F(s) + 2sF'(s) + 2F(s) + (\lambda - 1)F(s) = 0.$$

After a bit of algebra we obtain,

$$2sF'(s) + (s^2 + \lambda + 1)F(s) = 0$$

then,

$$F'(s) + \frac{(s^2 + \lambda + 1)}{2s}F(s) = 0.$$

This is a separable differential equation.

$$\frac{dF}{F} = -\frac{(s^2 + \lambda + 1)}{2s}ds$$

Which has a solution,

$$F = e^{\frac{-s^2}{4} - \frac{(\lambda+1)}{2}\ln(s) + C}$$

which simplifies to,

$$F = Ae^{\frac{-s^2}{4}}s^{\frac{-\lambda-1}{2}}.$$

Now our problem begins as we cannot invert this using the normal "read the table backwards method," we must use the Mellin Inversion Formula.

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} F(s)e^{st}ds.$$

Substituting our $F$ into the inversion formula, we obtain the highly complicated integral:

$$h(\xi) = \frac{A}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} s^{-\frac{(\lambda+1)}{2}} e^{\frac{-s^2}{4}} e^{s\xi} ds.$$

One possible solution to this is to set $\frac{(\lambda+1)}{2} = n + 1$ and we then obtain:

$$h(\xi) = \frac{A}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} s^{-(n+1)} e^{\frac{-s^2}{4} + s\xi} ds.$$

This is none other than the contour integral that generates Hermite Polynomials (see Appendix A.3). Thus we see the solutions for $h(\xi)$ are Hermite Polynomials, $H_n$, and we had a requirement that $\frac{(\lambda+1)}{2} = n + 1$. Thus our solution to the original problem is where we combine the two constants into A:

$$\psi = Ae^{\frac{-\xi^2}{2}} H_n.$$

We know that $\lambda = 2n + 1$ and we let $\lambda = \frac{2E}{\hbar}\sqrt{\frac{m}{k}}$ from earlier. We can now see that we have quantized energies. Knowing that the angular frequency is given by $\omega = \sqrt{\frac{k}{m}}$ So we see that $E_n$ is

$$E_n = (n + 1)\hbar\omega$$

While this way might be harder, we still see our quantized energy levels as Planck had found. Thus we found another analytic solution to the harmonic oscillator potential. Laplace Transforms are much harder to use in Quantum Mechanics as they carry no physical significance like a Fourier Transform, which is a transform that shifts from a time function to a frequency function.

*Chapter 5*

# Conclusion

In this project, we see how, from a simple calculus problem, we have gained ideas on how to make an accurate clock. To do that, we studied Laplace Transforms, an incredibly useful tool to solve differential equations. This ideal clock lead to improved longitude and time unification. That study of time and clocks lead Einstein and others to make leaps and bounds in the world of physics. Thus, we gained modern physics from where our classical understanding of objects and laws begins to break down. Finally, we see that Laplace Transforms can be useful in Quantum Mechanics as well. Through our understanding of time, we not only manage our daily lives, but also seek to describe the physical nature of nearly everything around us, and can prevent the death of the Pope's rhinoceros.

# Appendix A

# Appendix

In finding a solution to the Brachistochrone besides utilizing a plethora of calculus I used some trigonometric identities. In the following two sections I will derive the identities that I use, evaluate the Gaussian Integral and also discuss Hermite Polynomials which I utilized in the project.

## A.1 Gaussian Integral

Dealing with statistics and other fields in mathematics and the sciences we often run across the Gaussian integral.

$$\int_0^\infty e^{-x^2}\,dx$$

This integral can not be evaluated by normal means, so we must use a little trick. That integral is the area underneath the curve $e^{-x^2}$, so it is really just some number, I will call it G.

$$G = \int_0^\infty e^{-x^2}\,dx$$

To figure out what G is I will consider $G^2$ with the second integral using y as the variable.

$$G^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right)$$

Let us combine these integrals into a single double integral.

$$G^2 = \int_0^\infty \int_0^\infty e^{(-x^2-y^2)}\,dx\,dy$$

Let us convert to polar coordinates to evaluate this double integral. Recall polar coordinates are

$$r = \sqrt{x^2+y^2}$$

and

$$dxdy = rdrd\theta.$$

Our new integral in polar coordinates is:

$$G^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty re^{-r^2}\,dr\,d\theta.$$

We can easily separate the variables too

$$G^2 = \left(\int_0^{\frac{\pi}{2}} 1 d\theta\right)\left(\int_0^{\infty} re^{-r^2} dr\right).$$

The integral with respect to $\theta$ is simple, and the other can be solved with a simple substitution $u = -r^2$.

$$G^2 = \frac{\pi}{2}[-\frac{1}{2}(0-1)]$$

Thus

$$G^2 = \frac{\pi}{4}$$

$$G = \frac{1}{2}\sqrt{\pi}.$$

Finally, the original integral we sought is

$$G = \int_0^{\infty} e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi}.$$

A similar argument can be made for same integral over $\mathbb{R}$, with theta from $[0, 2\pi]$. Or we can observe that this integral is twice that of the other due to symmetry.

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}}$$

## A.2 Euler's Identity

Euler's formula is a powerful formula in mathematics that relates complex numbers to trigonometric functions. This formula has uses in mathematics, physics, complex analysis, and much more. To derive this formula we first need to recall some notable power series expansions and the fact that $i^2 = -1$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + ...$$

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}...$$

$$cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + ...$$

Now let us consider the expansion of $e^{ix}$

$$e^{(ix)} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + ...$$

We can turn power of $i$ into negative one and positive one and then separate terms with and without $i$ in them.

$$e^{(ix)} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)$$

Now observe that the first grouping is the power series expansion of cosine and the second grouping with the $i$ in front is the power series expansion of sine. Thus,

$$e^{(ix)} = cos(x) + isin(x).$$

And this famous formula is Euler's Formula. When we evaluate it for $x = \pi$ we get Euler's Identity.

$$\boxed{e^{i\pi} + 1 = 0}$$

This identity is a powerful equation in mathematics. It has 5 very important numbers: $0, 1, \pi, e$, and $i$; it also uses 3 common operations which we deal with: addition, multiplication, and exponentiation. Further let us square the formula.

$$(e^{ix})^2 = \big(cos(x) + isin(x)\big)^2 = cos^2(x) + 2icos(x)sin(x) - sin^2(x)$$

$$= \big(cos^2(x) - sin^2(x)\big) + 2icos(x)sin(x)$$

However, we also have

$$(e^{ix})^2 = e^{i2x} = cos(2x) + isin(2x)$$

If these are to be equal, that is if

$$cos(2x) + isin(2x) = \big(cos^2(x) - sin^2(x)\big) + 2icos(x)sin(x)$$

We must have the real parts being equal on both sides and the imaginary parts on both sides being equal.

$$\boxed{cos(2x) = cos^2(x) - sin^2(x)}$$

and

$$\boxed{sin(2x) = 2cos(x)sin(x)}$$

These two formulas are the double angle formulas! These renowned trigonometric identities can be derived from Euler's Formula. Specifically, we utilize these in solving the Brachistochrone and Tautochrone problems.

## A.3 Hermite Polynomials

Hermite Polynomials are named after French mathematician Charles Hermite, they were first discovered by Pierre-Simon Laplace, and they appear all over the world of mathematics. The Hermite Polynomials are a set of orthogonal polynomials and are complete. There are a few types of Hermite Polynomials, but the most useful form for a physicist is generated by the Rodrigues Formula:

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}.$$

Here are the first 5 Hermite Polynomials.

$$H_1 = 1$$

$$H_2 = 2x$$

$$H_3 = 4x^2 - 2$$

$$H_4 = 8x^3 - 12x$$

$$H_5 = 16x^4 - 48x^2 + 12$$

Hermite Polynomials are orthogonal, so that means for any two Hermite polynomials $n$ and $n'$ where $n \neq n'$:

$$\int_{-\infty}^{\infty} H_n(x) H_{n'}(x)\ dx = 0$$

Hermite Polynomials can be generated by the Hermite differential equation, with n being an integer:

$$H'' - 2xH' + 2nH = 0$$

or by the contour integral

$$\boxed{\int s^{-(n+1)} e^{\frac{-s^2}{4}+sx}\ ds.}$$

*Appendix B*

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